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Optimization of a metal honeycomb sandwich beam-bar subjected to torsion and bending

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Abstract

In this paper, we analyze a metal honeycomb sandwich beam/torsion bar subjected to combined loading conditions. The cell wall arrangement of the honeycomb core is addressed in the context of maximizing resistance to either bending, torsion, or combined bending and torsion for given dimensions, face sheet thicknesses and core relative density. It is found that the relative contributions of the honeycomb core to torsion and bending resistances are sensitive to the configuration of cell walls and the optimal properties significantly exceed those of stochastic metallic foams as sandwich beam core materials for this configuration.

Keywords: Metal honeycombs; Sandwich beam; Optimization; Torsion; Bending

1. Introduction

Cellular metals hold promise for applications as multifunctional materials in lightweight structures (Gibson and Ashby, 1997; Torquato et al., 1998; Ashby et al., 2000; Evans et al., 2001) that may require strength, stiffness, thermal insulation, wave scattering, mechanical energy absorption, and so on. Their properties are attractive for use as cores for beams, panels and other lightweight structures. In this paper, we considered ordered metallic cellular materials with extended prismatic cells, otherwise known as linear cellular alloys (LCAs) or metal honeycombs. As distinguished from traditional metal honeycombs, which have a characteristic periodic unit cell, typically hexagonal, the manufacturing process for LCAs enables tailoring of more complex in-plane morphologies of cell size and shape to achieve desired multi-functionality (Cochran et al., 2001).

As described in Figs. 1–3, we consider in this paper the behavior of a beam- or tube-like structure that can be extruded in a single step as a LCA (Cochran et al., 2001; Hayes et al., 2001). The face sheets on the

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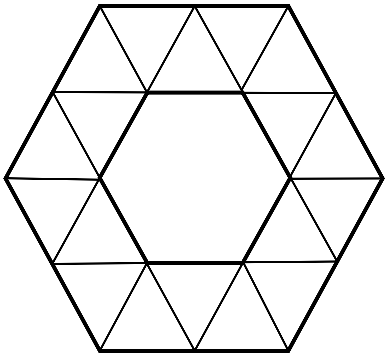

Fig. 1. A schematic hexagonal supercell.

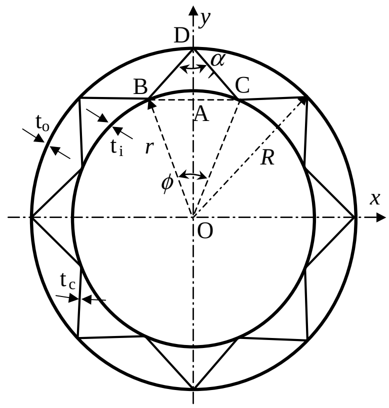

Fig. 2. A supercell with circle and triangle subcells.

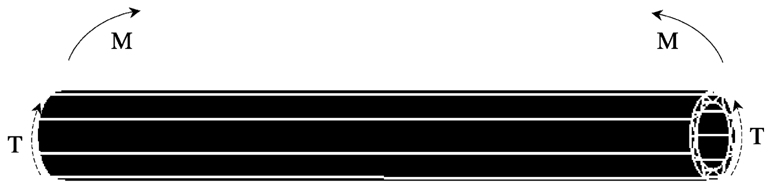

Fig. 3. A LCA sandwich beam with one supercell.

inner and outer radii can effectively be extruded as integral with the diagonal struts. Although such a beam-bar can serve as a conduit for fluid storage or flow as well, or as a heat exchanger device, we focus in this paper on purely mechanical torsion and bending behaviors for such a sandwich beam/torsion bar structure.

In particular, we focus on a circular bar with triangular subcells (Fig. 2). The equivalent torsional rigidity and bending rigidity of the sandwich bar structure are estimated using standard theoretical approaches. Prandtl's membrane analogy approach (Den Hartog, 1952) is employed to calculate the torsion resistance and standard beam theory (Gere and Timoshenko, 1984) is used to calculate the bending resistance. It will be shown that the contribution of the LCA core can be optimized by arranging the distribution of cell wall material. Comparisons between the LCAs and comparable stochastic foams as core materials within the same sandwich structures are presented. The contributions of the triangular, corrugated LCA core to torsional and bending rigidity are significantly higher than those of stochastic metal foam cores. Budiansky (1999) has analyzed a stochastic foam-filled member such as that shown in Fig. 3 subjected to compression along the tube axis in exploring minimum weight designs. The analysis approach taken here for in-plane behavior is similar to that of Kim and Kim (2000) in pursuing topology optimization of a 2-D cross-section of a beam.

2. Problem description

Fig. 3 shows the types of loading conditions to be considered on the LCA sandwich beam/torsion bar. In order to determine the respective contributions of the LCA core and outside and inside face sheets of the sandwich beam, its transverse cross-section is divided into three parts, as shown in Fig. 4. The parametric variables for describing the three parts—the core, the outside face sheet, and the inside face sheet—are respectively denoted by subscripts, e.g. $\square_c$, $\square_o$, and $\square_i$, where $\square$ is a generic variable. All face sheets and cell walls are assumed to be composed of the same material. The thicknesses of the three types of sheets/walls are respectively labeled as t_c, t_o and t_i, while the outside and inside radii of the cylinder LCA beam are respectively denoted by R and r. The angle between two adjacent cell walls in the core is α, as shown in Fig. 2.

According to simple theories of torsion and bending, maximum torsion and bending rigidities are achieved by arranging the mass as far as possible from the neutral axis or centerline of the cylindrical sandwich shaft. Hence, without imposition of constraints, the thickness of the outside face sheet, t_o, should be maximized at the expense of the thickness, t_c and t_i of the core cell walls and the inside face sheet. However, constraints typically exist by virtue of requirements for in-plane stiffness or transverse shear resistance, manufacturability, indentation resistance, face sheet yielding/buckling resistance, and other functional requirements such as heat or mass transport within the cells between face sheets. The thicknesses t_c and t_i should therefore be determined by meeting other needs or addressing potential failure modes for sandwich structures. For illustrative purposes, in this paper, the face sheet thicknesses t_o and t_i are assumed to be equal to each other, focusing attention on the distribution of cell walls within the LCA core to optimize rigidity under torsion, bending, or combined bending and torsion. The arrangement differs for each case, as might be expected.

The torsional resistance and second polar moment of area about the central axis of the beam receive contributions from each of the three parts of sandwich beam cross-section, i.e.,

$$J = J_o + J_c + J_i \quad \text{and} \quad I_p = I_o + I_c + I_i \tag{1}$$

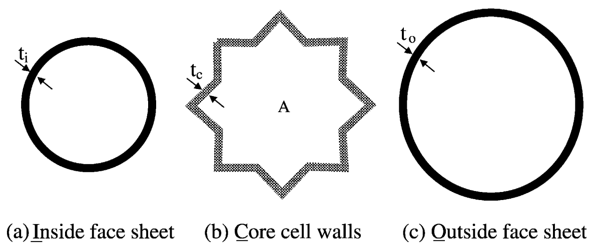

Fig. 4. Three parts of cross-section for LCA sandwich structure.

where $I_p = \int (x^2 + y^2)\,dA$; here, J is distinguished from I_p by virtue of the use of the Prandtl's membrane analogy approach to determine the former for the torsional case. For circular face sheets, $J_o \approx I_o$ and $J_i \approx I_i$, with increasing accuracy of the approximation of the membrane analogy for increasing R/t_o and r/t_i.

For the analysis of torsional and bending resistance of the sandwich beam, a representative unit cell (i.e., ΔBDC in Fig. 2) was extracted. In this case, only the cell walls BD and DC are considered as core cell walls since BC is the arc of the inner face sheet of the sandwich beam. The LCA core is composed of n sets of such a unit cell, where we will restrict n to even integers. The second polar moment of area of the core cell walls is given by

$$I_c = nI_{unit} \tag{2}$$

The torsional resistance constant for the LCA core, J_c, is evaluated by the summed length of all cell walls and the enclosed area based on the Prandtl's membrane analogy approach (Den Hartog, 1952).

The corresponding angle from the center of circle is designated as ϕ, which is directly related to the adjacent angle between two cell walls in a unit cell of the core, α, yielding

$$n\phi = 2\pi, \quad n = \mathrm{E\_INT}\left(\frac{2\pi}{\phi}\right) \tag{3}$$

where 'E_INT' represents the operation to obtain an even integer with no remainder.

In order to calculate the relative density of the LCA core, the total length of cell walls in the core is determined. First, the length of a single cell wall, BD, is calculated, related to the R, r and ϕ. By using the Pythagorean theorem in triangle ABD of Fig. 2,

$$\overline{\mathrm{BD}} = \sqrt{\overline{\mathrm{BA}}^2 + \overline{\mathrm{DA}}^2} = \sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}} \tag{4}$$

Introducing the above expression into Eq. (3), the total length of all the core cell walls, S_c, is given by

$$S_c = 2n\overline{\mathrm{BD}} = 2n\sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}} \tag{5a}$$

$$S_c = 2\frac{2\pi}{\phi}\overline{\mathrm{BD}} = \frac{4\pi}{\phi}\sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}} \tag{5b}$$

Finally, the relative density of LCA core is given by $\rho_c^* = \rho^*/\rho_s$, where ρ_s is the density of the cell wall material, is given to first order by

$$\rho_c^* = \frac{t_c S_c}{A_c} = \frac{2nt_c\sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}}}{\pi(R^2 - r^2)} \tag{6a}$$

$$\rho_c^* = \frac{t_c S_c}{A_c} = \frac{4t_c\sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}}}{\phi(R^2 - r^2)} \tag{6b}$$

where $A_c = \pi(R^2 - r^2)$ is the area enclosed between the outside and inside face sheets. We have assumed thin walls relative to the radii in writing this relative density expression.

3. Torsional and bending resistances

3.1. Torsional resistance

Prandtl's membrane analogy approach (Den Hartog, 1952) is used to evaluate the resistant constants of the entire cross-section. First, the torsional resistance of the LCA core cell walls is calculated as

$$J_c \approx \frac{1}{S_c} 4A^2 t_c \tag{7}$$

where A represents the area enclosed by all the LCA core cell walls. For the cell wall configuration in Figs. 2 and 4(b), A is given by

$$A = nS_{\mathrm{OBDC}} = n(S_{\mathrm{BDC}} + S_{\mathrm{OBC}}) = nRr\sin\frac{\phi}{2} = \frac{2\pi}{\phi}Rr\sin\frac{\phi}{2} \tag{8}$$

So J_c is given by

$$J_c = \frac{1}{2n\sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}}} 4\left(nRr\sin\frac{\phi}{2}\right)^2 t_c = \frac{4\pi R^2 r^2 \sin^2\left(\frac{\phi}{2}\right) t_c}{\phi\sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}}} \tag{9}$$

Substituting the expression for relative density of the LCA core, ρ_c^*, into Eq. (9), the resistance of the LCA core is obtained as

$$J_c = \frac{\pi R^2 r^2 (R^2 - r^2)\rho_c^* \left(\sin\frac{\phi}{2}\right)^2}{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}} \tag{10}$$

The torsional resistances of the inside and outside face sheets are readily calculated by the membrane analogy method as

$$J_i \approx \frac{1}{S_i} 4A_i^2 t_i = 2\pi r^3 t_i \tag{11}$$

$$J_o \approx \frac{1}{S_o} 4A_o^2 t_o = 2\pi R^3 t_o \tag{12}$$

Finally, the torsional resistance of the entire sandwich beam can be expressed as

$$J = J_o + J_c + J_i = 2\pi(r^3 t_i + R^3 t_o) + \frac{\pi R^2 r^2 (R^2 - r^2)\rho_c^* \sin^2\frac{\phi}{2}}{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}} \tag{13}$$

Since the thicknesses of the inside and outside face sheets are kept constant and the radii of the outside wall and inside wall are also set to certain values, for a given value of the relative density of the LCA core, ρ_c^*, it is obvious that optimal torsional resistance is achieved by the arrangement of LCA core cell walls, i.e., by the angle of one group unit cell walls as specified by either ϕ or α, as well as the thickness of the core cell edges.

For evaluating the optimal J, for a given core cell wall thickness, we calculate the derivative of J with respect to $\cos(\phi/2)$, i.e.,

$$\frac{\partial J}{\partial\left(\cos\frac{\phi}{2}\right)} = \frac{\partial J_c}{\partial\left(\cos\frac{\phi}{2}\right)} = \pi R^2 r^2 (R^2 - r^2)\rho_c^* \left[\frac{-2\cos\frac{\phi}{2}}{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}} - \frac{-2Rr\left(1 - \cos^2\frac{\phi}{2}\right)}{\left(R^2 + r^2 - 2Rr\cos\frac{\phi}{2}\right)^2}\right] \tag{14}$$

If let $\partial J/\partial(\cos(\phi/2)) = 0$, the extremal value of torsional resistance is obtained for

$$\cos\frac{\phi}{2} = \frac{r}{R}, \quad \phi = 2\cos^{-1}\left(\frac{r}{R}\right) \tag{15}$$

Hence, the torsional resistance of the LCA sandwich structure will approach the maximum value when $\phi = 2\cos^{-1}(r/R)$. The angle α between two adjacent cell walls in core can be determined from

$$r\sin\frac{\phi}{2} = \sin\frac{\alpha}{2}\sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}} \tag{16}$$

In the following, the angle ϕ will be used for convenience.

Normalizing both sides of Eq. (13) by r^2R^2, the expression of the torsional resistance of the LCA sandwich bar is rewritten as

$$\frac{J}{R^2r^2} = 2\pi\left(\frac{rt_{\mathrm{i}}}{R^2} + \frac{Rt_{\mathrm{o}}}{r^2}\right) + \frac{\pi\left(1 - \frac{r^2}{R^2}\right)\rho_{\mathrm{c}}^*\sin^2\frac{\phi}{2}}{1 + \frac{r^2}{R^2} - 2\frac{r}{R}\cos\frac{\phi}{2}} \tag{17}$$

where J/R^2r^2 is a non-dimensionalized quantity used to compare the relative magnitude of contribution of various structural elements to the overall torsional resistance. To more clearly understand the contributions of different parts of LCA beam cross-section, let $t_{\mathrm{i}} = t_{\mathrm{o}} = \frac{1}{10}(R - r)$, a reasonable value, and let the ratio r/R vary over the range $[0.7, 0.9]$, so a simple function is obtained as

$$\frac{J}{R^2r^2} = \frac{\pi}{5}\left(\frac{1}{x^2} + x\right)(1 - x) + \frac{\pi(1 - x^2)\rho_{\mathrm{c}}^*\sin^2\frac{\phi}{2}}{1 + x^2 - 2x\cos\frac{\phi}{2}} \tag{18}$$

where $x = r/R$.

3.2. Bending resistance

The bending resistance of the sandwich beam shown in Fig. 3 may be estimated using simple Euler–Bernouilli beam theory as

$$I_x = \int_A y^2\,\mathrm{d}A, \quad I_y = \int_A x^2\,\mathrm{d}A \tag{19}$$

Here, I_x and I_y are the second moments of area with respect to bending about the neutral axis in the X direction and the Y direction, respectively. The quantity $I_{\mathrm{p}} = \int_A (x^2 + y^2)\,\mathrm{d}A$ is related to I_x and I_y according to $I_{\mathrm{p}} \approx \frac{1}{2}I_x \approx \frac{1}{2}I_y$ due to the approximately circular symmetry of the entire cross-section of the LCA beam. Hence, for convenience, the polar second moment of area is mainly used to represent the bending resistance constant.

First, a representative cell wall BD is considered to analyze the bending resistance constant of the core cell walls as shown in Fig. 5. The second polar moment of area of the LCA core is given in terms of that of the segment BD by

$$I_{\mathrm{c}} = \sum_{i=1}^{n} 2I_{\mathrm{BD}} \tag{20}$$

It is noted that the current analysis is valid for the specific configuration of core walls when $n = 4, 8, 12, \ldots,$ $I_{\mathrm{p}}/2 = I_x = I_y$ due to the symmetry in both X and Y directions. For $n = 6, 10, 14\ldots,$ then $I_x \neq I_y$ but $I_{\mathrm{p}} = I_x + I_y$ still holds. For this case, the optimization of bending resistance might be performed with respect to each of the X and Y directions, but this is not within the scope of interest of the present paper.

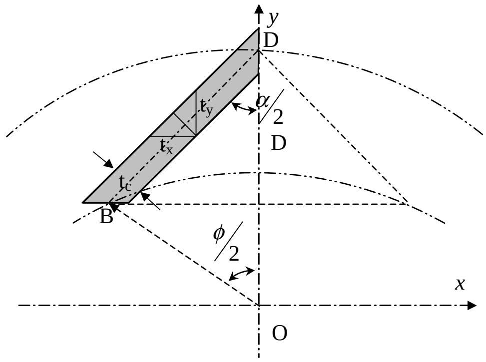

Fig. 5. A single representative cell wall for bending analysis.

The cell wall, BD, receives our attention. In Fig. 5, the geometric configuration has the relationships

$$t_x = \frac{t_c}{\cos\left(\frac{\alpha}{2}\right)}, \quad \cos\left(\frac{\alpha}{2}\right) = \frac{R - r\cos\left(\frac{\phi}{2}\right)}{\overline{\mathrm{BD}}}$$

$$t_y = \frac{t_c}{\sin\left(\frac{\alpha}{2}\right)}, \quad \sin\left(\frac{\alpha}{2}\right) = \frac{r\sin\left(\frac{\phi}{2}\right)}{\overline{\mathrm{BD}}}$$

where $\overline{\mathrm{BD}} = \sqrt{R^2 + r^2 - 2Rr\cos\frac{\phi}{2}}$. The second moment of area of the cell wall, BD, can be calculated for bending about the neutral axis in each of the X and Y directions as

$$(I_y)_{\mathrm{BD}} = \int_A (x^2)\,\mathrm{d}A = \int_{x_b}^{x_D} x^2\,\mathrm{d}(t_y x) = \frac{1}{3}t_y(x_D^3 - x_B^3), \quad x_D = 0, \quad x_B = -r\sin\left(\frac{\phi}{2}\right) \tag{21a}$$

$$(I_x)_{\mathrm{BD}} = \int_A (y^2)\,\mathrm{d}A = \int_{y_b}^{y_D} y^2\,\mathrm{d}(t_x y) = \frac{1}{3}t_x(y_D^3 - y_B^3), \quad y_D = R, \quad y_B = R - r\cos\left(\frac{\phi}{2}\right) \tag{21b}$$

Therefore, the polar moment of area of the LCA core is obtained as

$$I_{\mathrm{BD}} = (I_x)_{\mathrm{BD}} + (I_y)_{\mathrm{BD}} = \frac{1}{3}t_c\overline{\mathrm{BD}}\left(\frac{r^2R - r^3\cos(\frac{\phi}{2}) + 2Rr^2(\cos(\frac{\phi}{2}))^2 - 3R^2r\cos(\frac{\phi}{2})}{R - r\cos(\frac{\phi}{2})}\right) \tag{22}$$

According to Eq. (2), the bending resistance of the LCA core of the sandwich beam is given by

$$I_c = 2nI_{\mathrm{BD}} = \frac{2}{3}nt_c\overline{\mathrm{BD}}\left(\frac{2r^3(\cos(\frac{\phi}{2}))^3 - 4Rr^2(\cos(\frac{\phi}{2}))^2 + (3R^2r - r^3)\cos(\frac{\phi}{2}) + r^2R}{R - r\cos(\frac{\phi}{2})}\right) \tag{23}$$

Substituting the relative density, ρ_c^* of Eq. (6) into the Eq. (23) leads to

$$I_c = \frac{\pi}{3}\rho_c^*(R^2 - r^2)\left(\frac{2r^3(\cos(\frac{\phi}{2}))^3 - 4Rr^2(\cos(\frac{\phi}{2}))^2 + (3R^2r - r^3)\cos(\frac{\phi}{2}) + r^2R}{R - r\cos(\frac{\phi}{2})}\right) \tag{24}$$

The bending resistances of the inside and outside face sheets are readily estimated as

$$I_i = \frac{\pi}{2}\left[\left(r + \frac{t_i}{2}\right)^4 - \left(r - \frac{t_i}{2}\right)^4\right] \tag{25}$$

$$I_o = \frac{\pi}{2}\left[\left(R+\frac{t_o}{2}\right)^4 - \left(R-\frac{t_o}{2}\right)^4\right] \tag{26}$$

Finally, the bending resistance of the entire sandwich beam is obtained by applying Eq. (1), making use of Eqs. (24)–(26), i.e.,

$$I_p = I_o + I_c + I_i \tag{27}$$

As before, the thicknesses of the outside and inside face sheets are kept constant, and the radii of the outside and inside wall are also set to constants. The relative density of LCA core, ρ_c^*, is set to a certain value. So the optimal bending resistance of the sandwich beam is determined by varying with the cell wall arrangement, i.e., by the angle of one group unit cell wall, ϕ, as well as the core cell wall thickness.

To optimize for I_p, we determine the stationary point of the derivative of I_p with respect to $\cos(\frac{\phi}{2})$ in Eq. (27), i.e.,

$$\frac{\partial I_p}{\partial\left(\cos\frac{\phi}{2}\right)} = \frac{\partial I_c}{\partial\left(\cos\frac{\phi}{2}\right)} = \frac{\pi}{3}(R^2-r^2)\rho_c^*\left(\frac{-r^3+4Rr^2\cos\frac{\phi}{2}-3R^2r}{R-r\cos\frac{\phi}{2}} + \frac{r\left(r^2R-(r^3+3R^2r)\cos\frac{\phi}{2}+2Rr^2\left(\cos\frac{\phi}{2}\right)^2\right)}{\left(R-r\cos\frac{\phi}{2}\right)^2}\right) \tag{28}$$

Letting $\frac{\partial I_p}{\partial(\cos\frac{\phi}{2})} = 0$ and $|\cos\frac{\phi}{2}| \leqslant 1$, the extremal value of the bending resistance, I_p, is obtained when the cell-wall arrangement satisfies

$$\phi = 0 \tag{29}$$

This means that the maximum bending resistance is achieved when all the cell walls of LCA core radially connect the inside and outside face sheets. However, in this case the core cell walls are not arranged in triangular structure, and both transverse shear and torsional resistance are seriously compromised relative to the previous design. To obviate this, a small value of ϕ is required in the cell wall design, such that the bending resistance will approach the maximum value. However, this does not necessarily mean that the angle between two adjacent cell walls, α, is small. According to the geometric relationship in Fig. 3, the angle α can be much greater than ϕ for a certain ratio r/R.

Similarly, to more clearly understand the contributions of each of the three parts of the sandwich beam, let $t_i = t_o = \frac{1}{10}(R-r)$. Normalizing the bending resistance of the sandwich beam by R^2r^2 yields

$$\frac{I_p}{R^2r^2} = \frac{I_o+I_i}{R^2r^2} + \frac{I_c}{R^2r^2} \tag{30}$$

where the two components of right side of Eq. (30) are respectively expressed by

$$\frac{I_c}{R^2r^2} = \frac{\pi}{3}\rho_c^*(1-x^2)\left(\frac{2x(\cos(\frac{\phi}{2}))^3 - 4(\cos(\frac{\phi}{2}))^2 + \left(3\frac{1}{x}-x\right)\cos(\frac{\phi}{2}) + 1}{1-x\cos(\frac{\phi}{2})}\right),$$

$$\frac{I_i+I_o}{R^2r^2} = \frac{\pi x^2}{2}\left[\left(\frac{19}{20}+\frac{1}{20}\frac{1}{x}\right)^4 - \left(\frac{21}{20}-\frac{1}{20}\frac{1}{x}\right)^4\right] + \frac{\pi}{2x^2}\left[\left(\frac{21}{20}-\frac{1}{20}x\right)^4 - \left(\frac{19}{20}+\frac{1}{20}x\right)^4\right],$$

Here, $x = r/R$.

4. Results and discussion

In this section, the torsional rigidity and bending rigidity of the LCA sandwich structure are discussed for some special cases. Design for combined torsion and bending is studied. The contributions of torsional resistance and bending resistance of the LCA core are compared with the contributions of open- and closed cell stochastic metal foam core materials of the same relative density.

4.1. Torsional resistance

Based on the analytical estimates of torsional resistance of the LCA core sandwich structure, the maximum torsional resistance is obtained at $\phi = 2\cos^{-1}(r/R)$ for fixed r/R. If the ratio of r/R varies, such as $r/R = [0.7, 0.8, 0.9]$, a series of results are obtained. Curves of maximum torsional resistance for a specified core relative density of $\rho_c^* = 10\%$ versus the corresponding angle of arrangement, ϕ, are plotted in Fig. 6, where the thicknesses of the outside and inside face sheets are specified according to $t_i = t_o = 0.1(R - r)$.

Since the contributions of the inside and outside face sheets of the LCA sandwich structure to the torsional resistance are invariant with respect to the LCA core cell wall angle, the variation of the curves essentially reflect only the contribution of the LCA core with regard to optimization. In Fig. 6, every curve has a peak value of the torsional resistance when the angle of ϕ approaches the theoretical result $\phi = 2\cos^{-1}(r/R)$, while the negative values of angle ϕ express only redundant information. The optimal angle differs for various values of r/R, and the contribution of the LCA core greatly increases as the ratio of r/R decreases. From Fig. 6, the contribution of the LCA core tends toward zero as ϕ approaches zero. In this case the LCA core will not resist torsional loading because all cell walls are arranged radially.

To determine the proportion of the torsional resistance of the LCA core to the total torsional resistance of the sandwich structure, suppose $r/R = 0.8$ and let $\phi = 2\cos^{-1}(r/R)$, $t_i = t_o = 0.1(R - r)$, and $\rho_c^* = 10\%$; the components of torsional resistance are estimated by Eqs. (10)–(13) as

$$J_o + J_i = 0.2967r^2R^2, \quad J_c = 0.113r^2R^2 \quad \text{and} \quad J_c/J = 0.276 \tag{31}$$

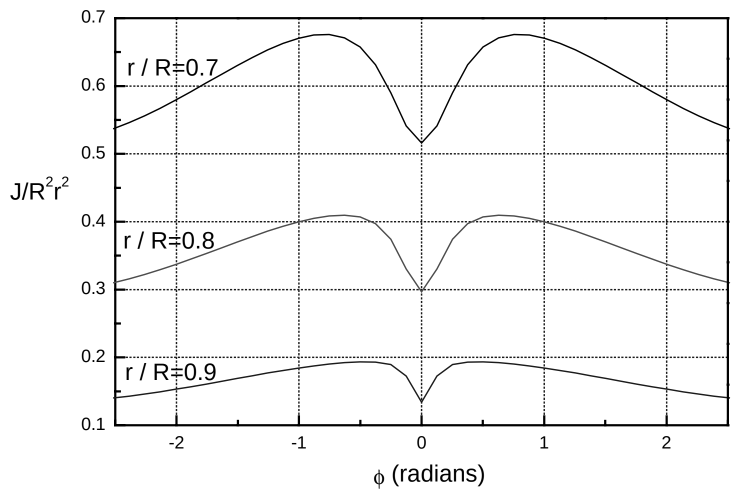

Fig. 6. Torsional resistance as a function of the arranged angle of LCA core cell walls, where $\rho_c^* = 10\%$ and the thickness of outside and inside faces are set to $t_i = t_o = 0.1(R - r)$.

Hence, the contribution of the LCA core is greater than 25% of the total torsional resistance, a very high value. If stochastic metal foam of the same relative density is used for the core of the sandwich beam, its contribution to torsional resistance is less than 5% in the following analysis. Its contribution was effectively considered as negligible by Evans et al. (2001). Clearly, the torsion resistance contribution of metal honeycomb cores plays an important role in sandwich structure design in contrast.

Next the torsional resistance of the sandwich structure core is compared to that of stochastic foam cores for the case where the thicknesses of the outside and inside face sheets are constant. The same relative densities for LCAs and for stochastic metal foams are assumed, i.e., $\rho_c^* = 0.1$. According to Eq. (10), the equivalent torsional resistance of the LCA core is given by

$$J_c G_s = \pi G_s R^2 r^2 \frac{[1-(r/R)^2]\rho_c^* \sin^2\frac{\phi}{2}}{1+(r/R)-2(r/R)\cos\frac{\phi}{2}} \tag{32}$$

where G_s is the shear modulus of the solid cell wall, i.e., $G_s = E_s/(2(1+\nu_s))$, and the Poisson's ratio of cell wall material is assigned as $\nu_s = 0.3$. Considering the limitation of core cell wall arrangement of $n = 4, 8, 12, \ldots$, an approximate optimal torsional resistance is obtained when $\phi = \pi/4$, $n = 8$, at $r/R = 0.8$. Eq. (32) is rewritten as

$$\frac{J_c G_s}{R^2 r^2} \approx 0.1464\pi G_s \frac{(1-(r/R)^2)\rho_c^*}{1+(r/R)^2-1.848(r/R)} \tag{33}$$

When a stochastic metallic foam is used as the core of the sandwich structure, the equivalent resistance of metal foam can be estimated according the results of the effective shear modulus for stochastic metal foams, including both open and closed cells (Gibson and Ashby, 1997). The equivalent torsional resistance of metal foam core of the sandwich beam can be estimated (see Eqs. A.1 and A.3 of Appendix A). For an open cell metal foam core, the equivalent torsional resistance is given by

$$J_c G_c^* = \frac{3}{16}\pi(R^4 - r^4)E_s(\rho_c^*)^2, \quad \text{or} \quad \frac{J_c G_c^*}{R^2 r^2} = \frac{3}{16}\pi\left(\frac{R^2}{r^2} - \frac{r^2}{R^2}\right)E_s(\rho_c^*)^2 \tag{34}$$

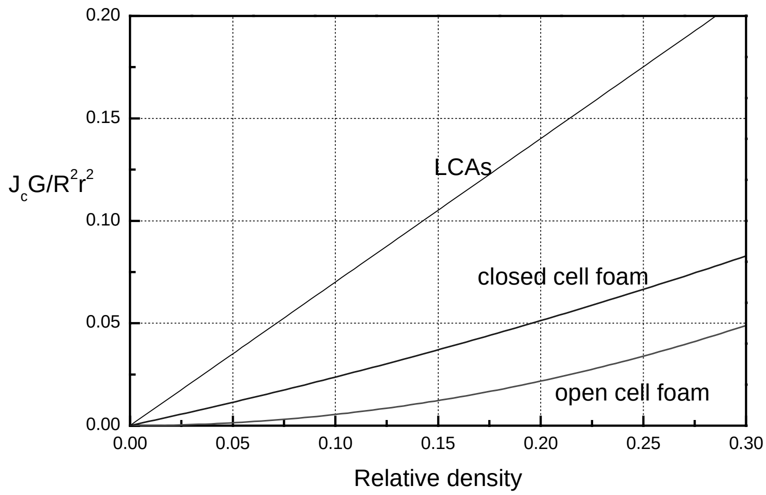

Fig. 7. Comparison of torsional resistance of an LCA core with metal foam cores of the same relative density.

For a closed cell metal foam core, the equivalent torsional resistance is given by

$$J_c G_c^* = \frac{3}{16}\pi(R^4 - r^4)E_s\{\omega^2(\rho_c^*)^2 + (1-\omega)\rho_c^*\},$$
$$\text{or } \frac{J_c G_c^*}{R^2 r^2} = \frac{3}{16}\pi\left(\frac{R^2}{r^2} - \frac{r^2}{R^2}\right)E_s\{\omega^2(\rho_c^*)^2 + (1-\omega)\rho_c^*\} \quad (35)$$

where let $\omega \approx 0.6$ for generic stochastic metal foams. Results of Eqs. (33)–(35) are compared in Fig. 7 for $E_s = 1$, without loss of generality. From Fig. 7, the torsional resistance contribution of the LCA core is higher than that of the stochastic metal foams. The contribution of a closed cell foam is a little higher than that of an open cell foam, but both are well below that of the LCA core.

4.2. Bending resistance

The analytical results of bending resistance of the LCA sandwich beam were presented in Eqs. (24)–(27) and (30). When different ratios of r/R are given, a series of bending resistance values are obtained. These results are plotted in Fig. 8, $r/R = [0.7, 0.8, 0.9]$, for the same relative density of the LCA core, $\rho_c^* = 0.10$.

From Fig. 8, the maximum bending resistance is obtained at $\phi = 0$, which is consistent with the analytical results of Eq. (29). In the different curves, the value of bending resistance near $\phi = 0$ become increasingly sensitive to variation of the angle as r/R increases. As mentioned previously, the angle ϕ should not be zero, so a small angle value is suggested in the realistic design and application. Recall that it does not mean the angle of α is very small between two adjacent cell walls.

When $r/R = 0.8$, the summation of bending resistance of the outside and inside face sheets is obtained by Eq. (30) as

$$I_o + I_i = 0.2967 r^2 R^2 \quad (36)$$

It is interesting that the value of the second term in Eq. (36) is the same as that of the torsional resistance in Eq. (31), which verifies the accuracy of the membrane analogy results for these geometries. Based on the latter result of $I_c > J_c$, the contribution of the LCA core to overall bending resistance is greater than that of the corresponding torsional resistance.

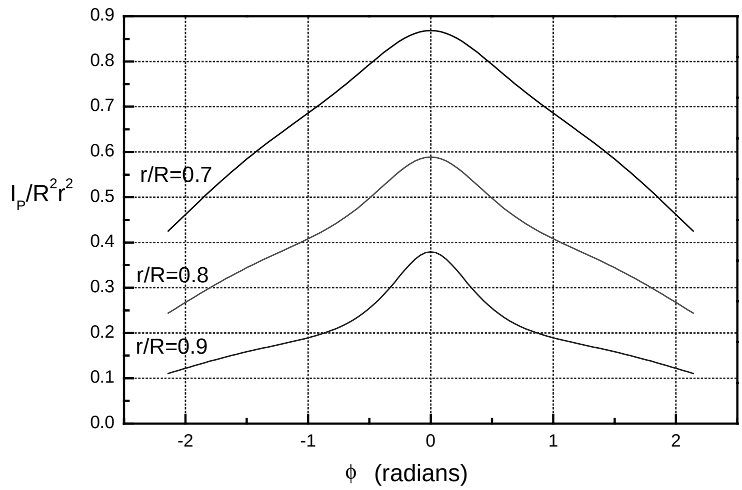

Fig. 8. Bending resistance variation with the arranged angle of LCA core cell walls.

The contribution of bending resistance of the LCA core to the overall sandwich beam can be compared with that of sandwich beams cores composed of stochastic metal foams (open cell foams or closed cell foams). Because the difference of bending resistance contribution originates from the different sandwich cores, the following discussion involves only the contribution of bending resistance of different sandwich cores. The same relative density for LCA and for stochastic metal foams is assumed, $\rho_c^* = 0.10$. The equivalent bending rigidities are normalized by R^2r^2 in the comparisons.

Recalling Eq. (30), the bending rigidity of the LCA core can be estimated by

$$\frac{I_c E_s}{R^2 r^2} = \frac{\pi}{3} E_s \rho_c^* (1 - (r/R)^2) \left(\frac{2(r/R)(\cos(\frac{\phi}{2}))^3 - 4(\cos(\frac{\phi}{2}))^2 + ((3R/r) - r/R)\cos(\frac{\phi}{2}) + 1}{1 - (r/R)\cos(\frac{\phi}{2})} \right) \tag{37}$$

Similarly, estimates of the effective Young's moduli of stochastic metal foams are available for open and closed cell foams (Gibson and Ashby, 1997). The equivalent bending rigidity of stochastic metal foams as a sandwich core can be estimated as follows. When using open cell metallic foam as the sandwich core, the contribution of the bending resistance is

$$I_c E_c^* = \frac{1}{2}\pi(R^4 - r^4)E_s(\rho_c^*)^2, \quad \text{or} \quad \frac{I_c E_c^*}{R^2 r^2} = \frac{1}{2}\pi\left(\frac{R^2}{r^2} - \frac{r^2}{R^2}\right)E_s(\rho_c^*)^2 \tag{38}$$

When using closed cell metallic foam as the sandwich core, the contribution to the bending resistance is

$$I_c E_c^* = \frac{1}{2}\pi(R^4 - r^4)E_s\{\omega^2(\rho_c^*)^2 + (1-\omega)\rho_c^*\} \tag{39a}$$

$$\frac{I_c E_c^*}{R^2 r^2} = \frac{1}{2}\pi\left(\frac{R^2}{r^2} - \frac{r^2}{R^2}\right)E_s\{\omega^2(\rho_c^*)^2 + (1-\omega)\rho_c^*\} \tag{39b}$$

Analogous to the torsion problem, let $\omega \approx 0.6$ for generic metal foams, and let $E_s = 1$ for the solid cell wall. For the LCA core, consider the ratio $r/R = 0.8$ and $\phi = \pi/4$. The results of Eqs. (37)–(39) are plotted with respect to the relative density in Fig. 9.

From Fig. 9, it is clear that the bending rigidity contribution of the LCA core is much higher than that of stochastic metal foams. The contribution of open cell foams is the lowest. This is because the bending

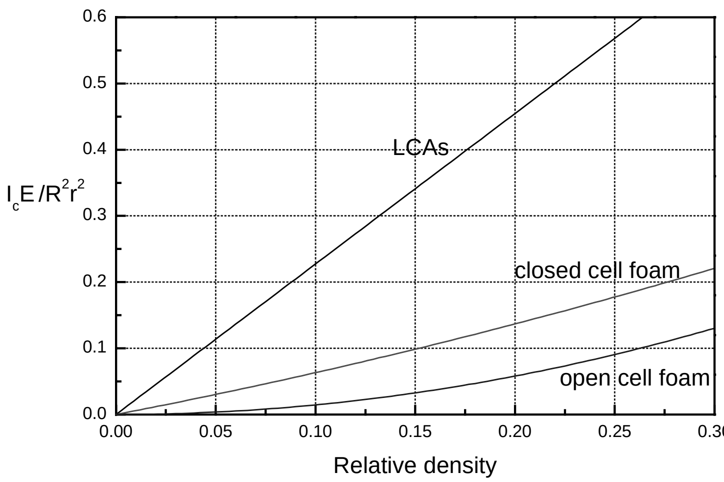

Fig. 9. Comparison of bending rigidity of LCA core with that of stochastic metal foam cores with respect to the relative density.

resistance contribution depends on both the cross-section geometry distribution and effective Young's modulus of the sandwich core. Proper arrangement of the cell walls of the LCA core can achieve higher bending stiffness than that of stochastic foams for the same relative density.

It must be pointed out that the torsional resistance and bending resistance of the LCA core, J_c and I_c, are not equal to each other. This is because that the core configuration of cell walls in the sandwich beam cross-section is not arranged with circular symmetry; hence $I_c > J_c$, which is demonstrated in previous results. When the sandwich core is made out of isotropic stochastic foams, a circular symmetrical cross-section is formed, so $J_c = I_c$. For the outside and inside face sheets, $J_o = I_o$ and $J_i = I_i$. As a matter of fact, in the present work of this paper, the thickness of t_o and t_i of outside and inside cell wall of LCA sandwich beam are kept constant, hence the torsional resistance and bending resistance of the overall sandwich beam only varies when the LCA core is modified or the r/R ratio is varied.

4.3. Optimization of combined torsion and bending

Next we consider the LCA sandwich beam structure subjected to combined torsion and bending. Weighing the relative importance of the torsion rigidity and bending rigidity is an issue that must be addressed based on the practical needs of each engineering application. A combined resistance constant is defined by

$$D = \omega_1 J + \omega_2 I_p, \quad \left(\sum \omega_i = 1, i = 1, 2\right) \tag{40}$$

where ω_1 and ω_2 are the weight coefficients for torsional resistance and bending resistance, respectively. In this paper, equal weighting is assumed, i.e., $\omega_1 = \omega_2 = 0.5$. Eq. (40) is normalized by the product R^2r^2, resulting in

$$\frac{D}{R^2r^2} = \frac{\omega_1 J}{R^2r^2} + \frac{\omega_2 I_p}{R^2r^2} \tag{41}$$

Substituting the results of Eqs. (17) and (30) into Eq. (41), the results of combined resistance are plotted in Fig. 10 for $t_i = t_o = 0.1(R - r)$ and $\rho_c^* = 10\%$.

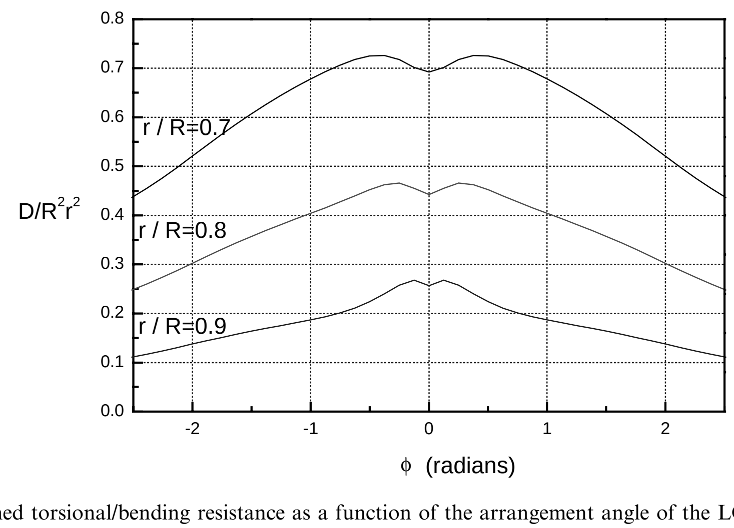

Fig. 10. Combined torsional/bending resistance as a function of the arrangement angle of the LCA core cell wall.

The combined resistance measure is plotted versus the angle ϕ for different ratios of r/R in Fig. 10. On each curve, two characteristic extremal points are found, one at $\phi = 0$ and the other one at $\phi \in [0, 2\cos^{-1}(r/R)]$. Solutions for only $\phi \geqslant 0$ are considered relevant by symmetry. The optimal angle should be near $\phi = 2\cos^{-1}(r/R)$. Another finding is that the shape of the curve at larger ratio r/R has stronger curvature near the optimal angle of LCA core cell walls because the values of torsional resistance and bending resistance exhibit stronger conflict such that a small change of the angle ϕ cause much larger change of the angle of α (Fig. 2) as the ratio r/R increases, i.e., the core width is becoming small and the spacing of face sheets decreases.

Finally, it is emphasized that optimization should be pursued for realistic sandwich structures that may include variable thickness of face sheets as well as the core cell walls. The analysis presented here can be readily extended to such cases but of course the parametric space that can be explored rises dramatically. This necessitates introduction of some practical manufacturing/application constraints to set bounds on the parameter space to be considered in the optimization exercise.

5. Summary and conclusions

Analyses were conducted for elastic torsional and bending resistance of a LCA sandwich beam-bar structure. The contribution of the LCA core was emphasized, and compared with those of stochastic open and closed cell metallic foams. The optimization of a LCA sandwich structure subjected to combined torsion and bending was discussed, based on the objective of obtaining maximum resistance to deformation for a given equal thickness of both face sheets, internal and external radii, and a specified relative density of the core (i.e., given mass).

Some key points are as follows:

(a) With the distribution and arrangement of LCA cell walls in a core with a given relative density, the optimal core contributions to overall torsional resistance and bending resistance, applied independently, were obtained, respectively, at $\phi = 2\cos^{-1}(r/R)$ and $\phi = 0$. This illustrates that the torsional and bending resistance of the LCA sandwich structure are in conflict under combined loading. The design of a LCA sandwich structure subjected to combined torsion and bending must be weighed according to the practical application requirements, and results in a different optimal included angle for the core cell walls.

(b) The contributions of the LCA core for the torsion and bending resistance constants for optimal stiffness is greater than 25% of the overall resistances for each case for a LCA core relative density of 10% and a thickness of outside and inside face sheets of $(R - r)/10$.

(c) Based on the comparisons of the relative contribution of the LCA core with those of stochastic metal foams to overall torsional and bending resistance of the sandwich beam structure, the LCA core is much more effective in contributing to overall stiffness, as the stochastic metal foams typically contribute less than 5% to resistance for low relative densities.

Clearly, the contribution of the LCA core should not be neglected in engineering design. It has been considered negligible in recent works on sandwich beam structures with stochastic foam cores. An extension of the present work to consider multiple layers of graded cells through the core of the sandwich beam structure might offer enhancements of optimal torsional and bending rigidities for a given relative density, especially concerning multifunctional optimization for combined torsion and bending.

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Appendix A. Stiffness of stochastic foams

For open cell foam core material, the effective elastic moduli are (Gibson and Ashby, 1997)

$$\frac{G_c^*}{E_s} \approx \frac{3}{8}(\rho_c^*)^2 \tag{A.1}$$

$$\frac{E_c^*}{E_s} \approx (\rho_c^*)^2 \tag{A.2}$$

where the subscript "c" denotes core material. For closed cell foam core material, the effective elastic moduli are (Gibson and Ashby, 1997)

$$\frac{G_c^*}{E_s} \approx \frac{3}{8}\{\omega^2(\rho_c^*)^2 + (1-\omega)\rho_c^*\} \tag{A.3}$$

$$\frac{E_c^*}{E_s} \approx \omega^2(\rho_c^*)^2 + (1-\omega)\rho_c^* \tag{A.4}$$

where $(1-\omega)$ is the fraction of solid which is contained in the cell faces. For most stochastic foams, ω is in the range of 0.6–0.8.

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